

Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 318 (2007) 1002–1026

**JOURNAL OF
Algebra**

www.elsevier.com/locate/jalgebra

Cohomology and deformations of the infinite-dimensional filiform Lie algebra \mathfrak{m}_0

Alice Fialowski ^{a,*}, Friedrich Wagemann ^b^a *Eötvös Loránd University, Budapest, Hungary*^b *Université de Nantes, Nantes, France*

Received 23 February 2007

Available online 29 October 2007

Communicated by Aner Shalev

Abstract

Denote \mathfrak{m}_0 the infinite-dimensional \mathbb{N} -graded Lie algebra defined by basis e_i , $i \geq 1$, and relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$. We compute in this article the bracket structure on $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$, $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$ and in relation to this, we establish that there are only finitely many true deformations of \mathfrak{m}_0 in each non-positive weight by constructing them explicitly. It turns out that in weight 0 one gets exactly the other two filiform Lie algebras.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Filiform Lie algebra; Cohomology; Deformation; Massey product

Introduction

Recall the classification of infinite-dimensional \mathbb{N} -graded Lie algebras $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ with one-dimensional homogeneous components \mathfrak{g}_i and two generators over a field of characteristic zero. A. Fialowski showed in [1] that any Lie algebra of this type, satisfying $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for every i , must be isomorphic to \mathfrak{m}_0 , \mathfrak{m}_2 or L_1 . We call these Lie algebras infinite-dimensional filiform Lie algebras in analogy with the finite-dimensional case where the name was coined by M. Vergne in [8]. Here \mathfrak{m}_0 is given by generators e_i , $i \geq 1$, and relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, \mathfrak{m}_2 with the same generators by relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$,

* Corresponding author.

E-mail addresses: fialowsk@cs.elte.hu (A. Fialowski), wagemann@math.univ-nantes.fr (F. Wagemann).

and L_1 with the same generators is given by the relations $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \geq 1$. L_1 appears as the positive part of the Witt algebra given by generators e_i for $i \in \mathbb{Z}$ with the same relations $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \in \mathbb{Z}$. The result was also obtained later by Shalev and Zelmanov in [7].

The cohomology with trivial coefficients of the Lie algebra L_1 was studied in [6], the adjoint cohomology in degrees 1, 2 and 3 has been computed in [2] and also all of its non-equivalent deformations were given. For the Lie algebra \mathfrak{m}_0 , the cohomology with trivial coefficients has been studied in [4], but neither the adjoint cohomology, nor related deformations have been computed so far. The reason is probably that—as happens usually for solvable Lie algebras—the cohomology is huge and therefore meaningless. Our point of view is that there still remain interesting features. We try to prove this in the present article by studying the adjoint cohomology of \mathfrak{m}_0 , while we reserve \mathfrak{m}_2 for a forthcoming paper.

Indeed, it is true that the first and second adjoint cohomology of \mathfrak{m}_0 are infinite-dimensional. The space $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ becomes already interesting when we split it up into homogeneous components $H_l^1(\mathfrak{m}_0, \mathfrak{m}_0)$ of weight $l \in \mathbb{Z}$, this latter space being finite-dimensional for each $l \in \mathbb{Z}$. We compute the bracket structure on $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ in Section 1.

The space $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$ is discussed in Section 2. This space is worse as it is infinite-dimensional even in each weight separately. The interesting new feature here is that there are only finitely many generators in each negative or zero weight which give rise to *true* deformations. Given a generator of $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$, i.e. an infinitesimal deformation, corresponding to the linear term of a formal deformation, one can try to adjust higher order terms in order to have the Jacobi identity in the deformed Lie algebra up to order k . If the Jacobi identity is satisfied to all orders, we will call it a true (formal) deformation, see Fuchs' book [5] for details on cohomology and [2] for deformations of Lie algebras.

In Section 3 we discuss Massey products, in Section 4 describe all true deformations in negative weights. Section 5 deals with deformations in zero and positive weights.

As obstructions to infinitesimal deformations given by classes in $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$ are expressed by Massey powers of these classes in $H^3(\mathfrak{m}_0, \mathfrak{m}_0)$, it is the vanishing of these Massey squares, cubes, etc., which selects within the $H_l^2(\mathfrak{m}_0, \mathfrak{m}_0)$ of weight l a finite number of cohomology classes. The main result reads

Theorem 1. *The true deformations of \mathfrak{m}_0 are finitely generated in each weight $l \leq 1$. More precisely, the space of unobstructed cohomology classes is in degree*

- $l \leq -3$ of dimension two,
- $l = 0$ of dimension two,
- $l = -2$ of dimension three,

while there is no true deformation in weight $l = -1$. In weight $l = 0$, these are deformations to \mathfrak{m}_2 and L_1 . In weight $l = 1$, there are exactly two true deformations, while in weight $l \geq 2$, there are at least two.

We do not have more precise information about how many true deformations there are in positive weight, but there are always at least two. As a deformation in these weights is a true deformation if and only if all of its Massey squares are zero (as cochains!), true deformations are determined by a countable infinite system of homogeneous quadratic equations in countably infinitely many variables. We did not succeed in determining the space of solutions of this system.

We believe that the discussion of these examples of deformations are interesting as they go beyond the usual approach where the condition that $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$ should be finite-dimensional is the starting point for the examination of deformations, namely the existence of a miniversal deformation [3].

Another attractive point of our study is the fact that in some cases the Massey squares and cubes involved are not zero because of general reasons, but because of the combinatorics of the relations. Thus the second adjoint cohomology of \mathfrak{m}_0 may serve as an example on which to study explicitly obstruction theory.

1. The space $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$

The Lie algebra \mathfrak{m}_0 is an \mathbb{N} -graded Lie algebra $\mathfrak{m}_0 = \bigoplus_{i=1}^{\infty} (\mathfrak{m}_0)_i$ with 1-dimensional graded components $(\mathfrak{m}_0)_i$ and generated in degree 1 and 2. Choosing a basis e_i of $(\mathfrak{m}_0)_i$, the only non-trivial brackets (up to skew-symmetry) read $[e_1, e_i] = e_{i+1}$ for all i . We are computing in this section the first cohomology space $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ of \mathfrak{m}_0 with adjoint coefficients. As Lie algebra and module are graded, the cohomology space splits up into homogeneous components, and we will always work with homogeneous cocycles $\omega(e_i) = a_i e_{i+l}$ for a scalar a_i and a given weight $l \in \mathbb{Z}$.

Concerning the cocycle identity $d\omega(e_j, e_i) = 0$, let us first suppose that $j = 1$ and $i > 1$ (up to choosing the symmetric case $j > 1$ and $i = 1$). In this case, it reads

$$\omega(e_{i+1}) = [e_1, \omega(e_i)] - [e_i, \omega(e_1)]$$

or, putting in the expression of ω , for all $l \geq 0$

$$a_{i+1}e_{i+l+1} = a_i e_{i+l+1} + \delta_{l,0} a_1 e_{i+l+1}.$$

This means that for all $l \geq 0$, we must have

$$a_{i+1} = a_i + \delta_{l,0} a_1,$$

while for $l = -1$, we get the previous equation for $i \geq 3$ and $a_3 = 0$, for $l = -2$, we get the previous equation for $i \geq 4$ and $a_4 = a_3 = 0$, and for $l \leq -3$, we get the previous equation for $i \geq -l + 2$ and $a_{-l+2} = a_{-l+1} = 0$, while there is no equation for $i < -l$.

The second situation where the cocycle identity has non-zero terms is when $l \leq -1$, and i and $j \geq 2$. In this case, there is only one non-zero term in the equation, and we get $a_i = 0$ for $i + l = 1$.

Now let us deduce the possible 1-cocycles in different weights:

Case 1. $l \leq -1$.

In case $l \leq -3$, the first identity means that all a_i for $i \geq -l + 2$ must be equal and $a_{-l+2} = a_{-l+1} = 0$, therefore all $a_i = 0$ for $i \geq -l + 1$, while there is no constraint on a_1, a_2, \dots, a_{-l} . This is compatible with the second situation.

In case $l = -1$ and $l = -2$, the first constraint implies that all $a_i = 0$ for $i \geq 3$, while there is no constraint on a_1 and a_2 . The second identity is then already satisfied for $l \leq -2$, while for $l = -1$, it implies $a_2 = 0$.

But observe that the formula $\omega(e_i) = a_i e_{i+l}$ makes sense for $l \leq -1$ only if $i \geq -l + 1$. Therefore all coefficients a_1, \dots, a_{-l} has to be set zero for $l \leq -1$.

In conclusion, all cohomology is zero in weight $l \leq -1$.

Case 2. $l \geq 1$.

In this case, the cocycle identity means that all a_i for $i \geq 2$ must be equal, while there is no constraint on a_1 .

Case 3. $l = 0$.

In this case, the first identity means that all a_i for $i \geq 3$ are determined by a_1 and a_2 , while there is no constraint on a_1 and a_2 .

Let us now examine the coboundaries: an element $x \in \mathfrak{m}_0$ determines a 1-coboundary by $\alpha_x(y) := [x, y]$ for all $y \in \mathfrak{m}_0$. In order to have a homogeneous coboundary, we must take $x = e_i$ for some $i > 0$; α_{e_i} is then homogeneous of weight i . Therefore we have:

- $dC_l^0(\mathfrak{m}_0, \mathfrak{m}_0) = 0$ for $l \leq 0$,
- $dC_l^0(\mathfrak{m}_0, \mathfrak{m}_0)$ generated by $de_l = [e_l, -]$ for $l \geq 1$.

Observe that the coboundaries for $l \geq 2$ are non-zero only on e_1 , thus they can modify only the a_1 -term of a cocycle. The coboundary for $l = 1$ is zero on e_1 and non-zero and constant on all other e_i . It thus kills the cocycle where all a_i for $i \geq 2$ are equal.

In conclusion, we have

Theorem 2.

$$\dim H_l^1(\mathfrak{m}_0, \mathfrak{m}_0) = \begin{cases} 1 & \text{for } l \geq 1, \\ 2 & \text{for } l = 0, \\ 0 & \text{for } l \leq -1. \end{cases}$$

Let us now determine representatives of the non-zero cohomology classes:

In $H_0^1(\mathfrak{m}_0, \mathfrak{m}_0)$, we have the generators ω_1 (corresponding to $a_1 = 1$ and $a_2 = 0$) and ω_2 (corresponding to $a_1 = 0$ and $a_2 = 1$) defined by:

$$\omega_1(e_k) = \begin{cases} e_1 & \text{for } k = 1, \\ 0 & \text{for } k = 2, \\ (k-2)e_k & \text{for } k \geq 3, \end{cases}$$

$$\omega_2(e_k) = \begin{cases} 0 & \text{for } k = 1, \\ e_k & \text{for } k \geq 2. \end{cases}$$

In $H_l^1(\mathfrak{m}_0, \mathfrak{m}_0)$ for $l \geq 1$, we have two different kinds of cocycles: there is γ for $l = 1$, and α_l for $l \geq 2$:

$$\gamma(e_k) = \begin{cases} ce_2 & \text{for } k = 1, \\ 0 & \text{for } k \geq 2, \end{cases}$$

$$\alpha_l(e_k) = \begin{cases} 0 & \text{for } k = 1, \\ b_l e_{k+l} & \text{for } k \geq 2. \end{cases}$$

It is well known that $H^*(\mathfrak{g}, \mathfrak{g})$ carries a graded Lie algebra structure for any Lie algebra \mathfrak{g} , and that $H^1(\mathfrak{g}, \mathfrak{g})$ forms a graded Lie subalgebra. Let us compute this bracket structure on our generators:

Given $a \in C^p(\mathfrak{g}, \mathfrak{g})$ and $b \in C^q(\mathfrak{g}, \mathfrak{g})$, define

$$ab(x_1, \dots, x_{p+q-1}) = \sum_{\sigma \in \text{Sh}_{p,q}} (-1)^{\text{sgn } \sigma} a(b(x_{i_1}, \dots, x_{i_q}), x_{j_1}, \dots, x_{j_{p-1}})$$

for $x_1, \dots, x_{p+q-1} \in \mathfrak{g}$. The bracket is then defined by

$$[a, b] = ab - (-1)^{(p-1)(q-1)} ba.$$

It thus reads on $H^1(\mathfrak{g}, \mathfrak{g})$ simply

$$[a, b](x) = a(b(x)) - b(a(x)).$$

We compute

$$[\omega_1, \alpha_l](e_k) = \begin{cases} 0 & \text{for } k = 1, \\ b_l(2+l-2)e_{2+l} & \text{for } k = 2, \\ (k+l-2)b_l e_{k+l} - (k-2)b_l e_{k+l} & \text{for } k \geq 3. \end{cases}$$

Therefore $[\omega_1, \alpha_l] = l\alpha_l$,

$$[\omega_1, \gamma](e_k) = \omega_1(\delta_{k1}ce_2) - \gamma(\delta_{k1}e_1).$$

Therefore $[\omega_1, \gamma] = -\gamma$,

$$[\omega_2, \alpha_l](e_k) = \begin{cases} 0 & \text{for } k = 1, \\ \omega_2(b_l e_{k+l}) - \alpha_l(e_k) = 0 & \text{for } k \geq 2. \end{cases}$$

Therefore $[\omega_2, \alpha_l] = 0$,

$$[\omega_2, \gamma](e_k) = \begin{cases} \omega_2(ce_2) - 0 & \text{for } k = 1, \\ 0 - \gamma(e_k) = 0 & \text{for } k \geq 2. \end{cases}$$

Therefore $[\omega_2, \gamma] = \gamma$,

$$[\alpha_l, \gamma](e_k) = \begin{cases} \alpha_l(ce_2) - 0 & \text{for } k = 1, \\ 0 - \gamma(b_l e_{k+l}) = 0 & \text{for } k \geq 2. \end{cases}$$

This gives $[\alpha_l, \gamma] = \delta_{k1} c b_l e_{2+l}$. This is a cocycle in weight $l + 1$, $l \geq 2$, but by the list of coboundaries in weight ≥ 2 , we see that it is actually a coboundary. Therefore we have $[\alpha_l, \gamma] = 0$ in cohomology.

$$[\omega_1, \omega_2](e_k) = \begin{cases} 0 - 0 = 0 & \text{for } k = 1, \\ 0 - 0 = 0 & \text{for } k = 2, \\ (k - 2)e_k - (k - 2)e_k = 0 & \text{for } k \geq 3. \end{cases}$$

Therefore $[\omega_1, \omega_2] = 0$. It is also rather clear that $[\alpha_l, \alpha_m] = 0$.

In summary:

Theorem 3. *The bracket structure on $H^1(\mathfrak{m}_0, \mathfrak{m}_0)$ is described as follows: the commuting weight zero generators ω_1 and ω_2 act on the trivial Lie algebra generated by γ in weight 1 and the α_l for weight $l \geq 2$ as grading elements, γ has degree -1 with respect to ω_1 , degree 1 with respect to ω_2 , while α_l has degree l with respect to ω_1 and degree 0 with respect to ω_2 .*

2. The space $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$

Let us first compute $H^2(\mathfrak{m}_0, \mathfrak{m}_0)$. We work with homogeneous cocycles $\omega(e_i, e_j) = a_{ij}e_{i+j+l}$ for a fixed weight $l \in \mathbb{Z}$, and for $i, j \geq 1$, $i \neq j$.

2.0. Observe that for weights $l \leq -3$, there are forbidden coefficients $a_{i,j}$, because they show up in front of e_{i+j+l} with $i + j + l \leq 0$. For example in $l = -3$, $a_{1,2}$ must be set to zero, in weight $l = -4$, $a_{1,2}$, $a_{1,3}$ must be set to zero, and so on.

2.1. The cocycle identity reads

$$\begin{aligned} d\omega(e_i, e_j, e_k) &= \omega([e_i, e_j], e_k) + \omega([e_j, e_k], e_i) + \omega([e_k, e_i], e_j) \\ &\quad - [e_i, \omega(e_j, e_k)] - [e_j, \omega(e_k, e_i)] - [e_k, \omega(e_i, e_j)] = 0. \end{aligned}$$

Let us first suppose that one index is equal to 1. The identity reads then

$$(a_{i+1,j} + a_{i,j+1})e_{i+j+l+1} = (a_{i,j} - \delta_{j+l,0}a_{j,1} - \delta_{i+l,0}a_{1,i})e_{i+j+l+1}, \quad (1)$$

where $i, j \geq 2$, $i \neq j$. This identity makes only sense for $i + j + l \geq 2$, because in the above equation, the $a_{i,j}$ term shows up in front of the bracket of e_1 with e_{i+j+l} . It is therefore not valid uniformly for all i, j starting from $l \leq -4$.

For $i + j + l < 0$, there is no equation, while there is a special equation for $i + j + l = 0, 1$, namely

$$a_{i+1,j} + a_{i,j+1} = 0.$$

Note that by 2.0, coefficients $a_{i,j}$ with $i + j + l \leq 0$ are set to zero.

2.2. The cocycle identity for e_i, e_j, e_k for $i, j, k \geq 2, i \neq j, i \neq k$, and $j \neq k$, gives a non-zero factor only if $j + k + l = 1$ or $i + j + l = 1$ or $i + k + l = 1$ (thus for $l \leq -4$). One can always arrange that only one factor is possibly non-zero for given i, j with $i + j + l = 1$ (by choosing $k = \max(i + 1, j + 1)$, for example). Thus for weight $l \leq -4$, the coefficient $a_{i,j}$ with $i + j = -l + 1, i \neq j, i, j \geq 2$, must be zero (which is compatible with the special equation!).

2.3. Let us now consider coboundaries: expressing that $\omega \in Z_l^2(\mathfrak{m}_0, \mathfrak{m}_0)$ is a coboundary $\omega = d\alpha$ for some 1-cochain $\alpha \in C_l^1(\mathfrak{m}_0, \mathfrak{m}_0)$, $\alpha(e_i) = \alpha_i e_{i+l}$ for all $i \geq 1$, gives by evaluation on e_i and e_j :

$$a_{i,j} e_{i+j+l} = \alpha([e_i, e_j]) - [e_i, \alpha(e_j)] + [e_j, \alpha(e_i)].$$

This equation makes sense only for $i + j + l \geq 1$ as all terms are multiples of e_{i+j+l} . Let us first take one index to be 1, then we get

$$a_{1,i} = \alpha_{i+1} - \alpha_i + \alpha_1 \delta_{l,0}$$

for all $i \geq \max(2, -l + 2)$, because α_i appears in front of the bracket of e_1 with e_{l+i} . Thus all $a_{1,i}, i \geq \max(2, -l + 2)$, can be taken to be zero by adding a coboundary. For $i = -l, -l + 1 \geq 2$, we have the special equation

$$a_{1,i} = \alpha_{i+1}.$$

It is now clear that, up to a coboundary, we may suppose for any l that the last two terms in Eq. (1) are zero. Observe that the non-coboundary terms $a_{1,i}$ for $l \leq -3$ in a general cocycle, namely the terms with $i = 2, \dots, -l - 1$, must be set to zero by 2.0.

2.4. For weight $l \leq -1$, we have additional coboundaries: indeed, there is a non-zero term in the coboundary equation for $e_i, e_j, i, j \geq 2, i \neq j$ yielding

$$a_{-l+1,j} = -\alpha_{-l+1}$$

for all $j \geq 2, j \neq -l + 1$. Be aware that the coefficient α_{-l+1} of the coboundary $d\alpha$ is linked to $a_{1,-l}$ by the equation $\alpha_{-l+1} = a_{1,-l}$ (cf. 2.3). Thus we cannot choose at the same time to render $a_{1,-l} = 0$ and $a_{-l+1,j} = 0$ in weight $l \leq -1$ by addition of a coboundary, we can impose only one of these conditions. This means for example that the cocycle given by coefficients $a_{i,j}$ with $a_{2,j} = 1$ for all $j \geq 3$ and $a_{i,j} = 0$ for all $i, j \neq 2$ (“the 2-family,” cf. 2.5) is a coboundary in weight $l = -1$. Here $a_{1,1} = 0$ and α_2 are not linked. More generally, the cochain given by coefficients $a_{i,j}$ with $a_{m+1,j} = 1$ for all $j \geq m + 2, a_{i,j} = 0$ for all other $i, j > m + 1$ (unless those which must be non-zero in order to respect antisymmetry) is cohomologous to the cocycle consisting of the only non-trivial coefficient $a_{1,m} = 1$ in weight $l = -m \leq -2$.

Let us now reconsider Eq. (1) in the stable range, i.e. with i and j such that $i, j \geq 2, i \neq j$, and $i + j \geq -l + 2$:

$$a_{i+1,j} + a_{i,j+1} = a_{ij}. \quad (2)$$

We will adopt two different points of view on this system of equations:

2.5. First point of view

Call the equations $a_{2,3} = a_{2,4}$, $a_{3,4} = a_{3,5}$, $a_{4,5} = a_{4,6}$, \dots , *diagonal equations*, and the terms involved *diagonal terms*. The prescription

$$a_{i,i+1} = a_{i,i+2} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k \end{cases}$$

specifies uniquely (unicity is shown by induction) a solution to this system, called the k th *family* or k -series. For the k th family, all $a_{r,s}$ with $r > k$ are zero, $a_{r,s} = 1$ for $r = k$, $a_{r,s}$ is linear in k for $r = k - 1$ (and s sufficiently big), $a_{r,s}$ is quadratic in k for $r = k - 2$ (and s sufficiently big), and so on.

Let us consider some examples, while we refer to Sections 4.5 and 5.1 for more informations; in the following expressions, all coefficients involving an index 1 are set to zero, and the first non-zero column starting from the RHS (i.e. the non-zero elements of the column $\{a_{m,k}\}_{k \geq m+1}$) is normalized to 1.

The 2 family. $a_{2,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 3$.

The 3 family. $a_{3,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 4$, and $a_{2,3} = a_{2,4} = 0$, $a_{2,k} = -(k - 4)$ for all $k \geq 5$.

The 4 family. $a_{4,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 5$, $a_{3,4} = a_{3,5} = 0$, $a_{3,k} = -(k - 5)$ for all $k \geq 6$, and $a_{2,3} = a_{2,4} = a_{2,5} = a_{2,6} = 0$, $a_{2,k} = \frac{(k-5)(k-6)}{2}$ for all $k \geq 7$ (even for all $k \geq 5$).

The 5 family. $a_{5,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 6$, $a_{4,5} = a_{4,6} = 0$, $a_{4,k} = -(k - 6)$ for all $k \geq 7$, $a_{3,4} = a_{3,5} = a_{3,6} = a_{3,7} = 0$, $a_{3,k} = \frac{(k-6)(k-7)}{2}$ for all $k \geq 8$ (even for all $k \geq 6$), and $a_{2,3} = a_{2,4} = a_{2,5} = a_{2,6} = a_{2,7} = a_{2,8} = 0$, $a_{2,k} = -\frac{(k-6)(k-7)(k-8)}{3!}$ for all $k \geq 9$ (even for all $k \geq 6$).

2.6. Second point of view

One can specify $a_{2,n}$ for all n , in such a way that the diagonal equations are satisfied. This implies that by choosing pairs $(a_{2,3}, a_{2,4})$, $(a_{2,5}, a_{2,6})$, $(a_{2,7}, a_{2,8})$, and so on, the first member is free, while the second member is determined by the corresponding diagonal equation.

Indeed, in $(a_{2,3}, a_{2,4})$, $a_{2,4}$ is determined by $a_{2,3} = a_{2,4}$, in $(a_{2,5}, a_{2,6})$, $a_{2,6}$ is determined by $a_{2,5} - a_{2,4} = a_{2,6} - a_{2,5}$ (which is just $a_{3,4} = a_{3,5}$), in $(a_{2,7}, a_{2,8})$, $a_{2,8}$ is determined by $((a_{2,7} - a_{2,6}) - (a_{2,5} - a_{2,4})) = ((a_{2,3} - a_{2,7}) - (a_{2,7} - a_{2,6}))$ (which is just $a_{4,5} = a_{4,6}$). All the other coefficients are then uniquely determined.

2.7. In conclusion, it is clear that for each weight $l \in \mathbb{Z}$, there is a countably infinite number of independent 2-cohomology classes. More precisely, in weight $l > -4$, the k families with $k = 2, 3, \dots$ represent independent 2-cohomology classes. In weight $l \leq -4$, 2.2 shows that the k family is contradictory for

$$k < \begin{cases} 2 + \frac{-l-3}{2} & \text{for } l \text{ odd,} \\ 2 + \frac{-l-2}{2} & \text{for } l \text{ even.} \end{cases}$$

But there is still a countably infinite number of independent 2-cohomology classes in each weight.

Theorem 4.

$$\dim H_l^2(\mathfrak{m}_0, \mathfrak{m}_0) = \infty$$

for each weight $l \in \mathbb{Z}$.

3. Massey products and deformations

The 2-cohomology is rather meaningless, as it is infinite-dimensional even in each weight separately. We ask now which of these homogeneous 2-cocycles gives rise to a deformation of \mathfrak{m}_0 . A necessary condition is that the class of the Massey square of the cocycle in question is zero. The first thing we will show is that for a large range of weights, even the condition that the Massey square is zero as a cochain is necessary and sufficient, and we will then determine all 2-cocycles which have zero Massey square and are thus the infinitesimal part of a deformation of \mathfrak{m}_0 which is polynomial and of polynomial degree 1. We will show in Section 5 that \mathfrak{m}_0 deforms (in a homogeneous way) to \mathfrak{m}_2 and to L_1 , but to no other \mathbb{N} -graded Lie algebra (non-isomorphic to \mathfrak{m}_0 , \mathfrak{m}_2 , and L_1). This is consistent with the classification of \mathbb{N} -graded Lie algebras with 1-dimensional graded components, generated in degrees 1 and 2 [1].

Let ω be a 2-cocycle, given as above by its coefficients $a_{i,j}$. The *Massey square* of ω is by definition

$$M(a)_{ijk} = (a_{i,j}a_{i+j+l,k} + a_{j,k}a_{j+k+l,i} + a_{k,i}a_{k+i+l,j})e_{i+j+k+2l}. \quad (3)$$

Observe that $M(a)_{ijk} = 0$ if any two indices coincide.

3.1. Massey squares and deformations

Proposition 1. *Let $l \geq -1$. If there exist i, j, k with $M(a)_{ijk} \neq 0$, then it is a non-trivial 3-cohomology class, and the 2-cocycle ω is obstructed. Thus ω is obstructed if and only if there exist i, j, k with $M(a)_{ijk} \neq 0$.*

Proof. Let $\alpha \in C^2(\mathfrak{m}_0, \mathfrak{m}_0)$ be a homogeneous 2-cochain with $\alpha(e_i, e_j) = b_{i,j}e_{i+j+m}$. Then

$$d\alpha(e_i, e_j, e_k) = \alpha([e_i, e_j], e_k) - [e_i, \alpha(e_j, e_k)] + \text{cycl.}$$

Given a 2-cocycle ω with non-zero Massey square $M(a)_{ijk}$, one wants to find α which compensates $M(a)_{ijk}$, i.e. with $d\alpha(e_i, e_j, e_k) = M(a)_{ijk}$. As $M(a)_{ijk}$ is of weight $2l$, one must have $m = 2l$.

For $i, j, k \geq 2$, there is only one non-zero term in the coboundary equation (cf. 2.4), and we have $d\alpha(e_i, e_j, e_k) = b_{i,j}e_{k+1}$ in case $i + j + 2l = 1$ (the cases $j + k + 2l = 1$ or $i + k + 2l = 1$ are similar). Thus we can compensate all Massey squares $M(a)_{ijk}$ with $i + j + 2l = 1$, $j + k + 2l = 1$ or $i + k + 2l = 1$. As $i, j, k \geq 2$, the highest weight case appears for $l = -2$.

On the other hand, for $l \geq -1$ all $a_{1,i}$ can be taken to be zero by adding coboundaries (cf. 2.3). Thus $M(a)_{ijk} = 0$ if one index is equal to 1, and the only squares to compensate are those with $i, j, k \geq 2$. \square

An interesting fact to note from the above proof is that the Massey squares that one can compensate by 3-coboundaries are the $M(a)_{ijk}$ with $i + j + 2l = 1$, $j + k + 2l = 1$ or $i + k + 2l = 1$ for $i, j, k \geq 2$ in weight $l \leq -2$. In the following, we will use the notation M_{ijk} for the coefficient of $e_{i+j+k+2l}$ in the corresponding Massey square.

4. Deformations in negative weights

4.1. True deformations in weight -1

We now consider only *square zero cohomology* in weight -1 , i.e. those classes in $H_{-1}^2(m_0, m_0)$ with Massey square equal to zero (not only as a cohomology class, but as a cochain!). By the previous section, this determines all deformations of m_0 in weight -1 .

First of all, the 2-family (cf. 2.5) is a square zero 2-cocycle, and it is not contradictory in weight -1 (cf. 2.7). But the 2-family is actually a coboundary according to 2.4.

Now assume that $a_{i,j}$ for all $i, j \geq 2$, $i \neq j$, defines a normalized cocycle, i.e. $a_{1,s} = 0$ for all s , which we may assume according to 2.3. Observe that for $l = -1$, there is no special equation of type $a_{1,-l} = \alpha_{-l+1}$. We assume further that all Massey squares M_{ijk} are zero. Suppose that k is the first integer such that $a_{3,k} \neq 0$, $k \geq 4$.

As all 3 coefficients below $a_{3,k}$ are zero, Eq. (2) shows that the first non-zero 4 coefficient is $a_{4,k-1}$, and that all 2 coefficients are equal up to $a_{2,k}$, while $a_{2,k} \neq a_{2,k+1}$.

Denote $a_{2,3} = c$. We will establish a table for the coefficients in order to examine the possible cases:

Lemma 1. $a_{3,k} \neq 0$ implies $a_{3,k+1} \neq 0$.

Proof. If $a_{3,k+1} = 0$, then $M_{23k} = a_{2,3}a_{4,k} + a_{3,k}a_{k+2,2}$. But by Eq. (2), $a_{3,k} = a_{4,k}$ and $a_{2,k} - a_{3,k} = a_{2,k+1} = a_{2,k+2}$, and therefore $M_{23k} = a_{2,3}a_{3,k} - a_{3,k}(a_{2,3} - a_{3,k}) = a_{3,k}^2 \neq 0$. This contradiction shows that $a_{3,k+1} \neq 0$. \square

For $k = 4$, $a_{3,k} = a_{3,k+1}$. Then $M_{234} = a_{3,4}(a_{2,4} - a_{2,6})$. If $a := a_{3,4} \neq 0$, then $a_{2,4} = a_{2,6} =: c$, and we have $a_{2,5} = c - a$ and thus $a - c = a + c$, implying $a = 0$: contradiction.

Let us now suppose $k > 5$. Then $M_{34(k-1)} = a_{4,k-1}a_{k+2,3} = 0$ implies $a_{k+2,3} = 0$, because $a_{4,k-1} = -a_{3,k} \neq 0$ by Eq. (2). Consideration of $M_{23(k+1)} = 0$ gives $a_{2,3} = a_{2,k+2} = a_{2,k+3}$. $M_{23(k+2)} = 0$ gives $a_{4,k+2} = -a_{3,k+3}$. $M_{34k} = 0$ implies that either $a_{3,k+3} = 0$ or $a_{3,k+1} = 2a_{3,k}$. But in this last case, $a_{5,k-1} = -4a_{3,k}$, and $M_{35(k-1)} = 0$ gives $a_{3,k+3} = 0$ anyhow. $M_{34(k+1)} = 0$ gives $a_{3,k+4} = a_{4,k+3}$.

This gives the following table for the coefficients $a_{i,j}$ (where we used i as the column index and j as the row index, contrary to the usual convention) with $a_{3,k} = a$ and $a_{3,k+1} = b$:

	2	3	4	5	6
$k-3$	c	0	0	0	$-a$
$k-2$	c	0	0	a	$3a+b$
$k-1$	c	0	$-a$	$-(2a+b)$	$-3a+b$
k	c	a	$a-b$	$a-2b$	$a-3b$
$k+1$	$c+b$	b	b	b	b
$k+2$	c	0	0	0	0

But the relation $a_{2,k} = c = a_{2,k+1} + a_{3,k} = c + b + a$ shows that $a = -b$.

Now, if k is odd, we have $a_{3,k} = a$, $a_{5,k-2} = a$, $a_{7,k-4} = a$, and so on, until we reach the diagonal $a_{j,j} = a = 0$. Thus in this case we have a contradiction.

But if k is even, we will take the line of $a_{i,j}$ given by $i + j = 4 + k$, and go to the diagonal: finally, we will also get $a = 0$, i.e. a contradiction.

In conclusion, a non-zero 3 coefficient for a square zero cocycle leads in weight -1 to a contradiction. But then all 2 coefficients must be equal by Eq. (2), and this gives the 2 family.

In conclusion, we have shown

Proposition 2. *There is no non-trivial square-zero cohomology class in weight $l = -1$. In particular, there does not exist any non-trivial true cohomology class in weight $l = -1$.*

4.2. True deformations in weight -2

We saw in the last section how to determine all cocycles, here in weight -2 , which lead to true deformations. But as proposition 1 in Section 3.1 is not valid in weight -2 , we cannot use the vanishing of all Massey squares to get restrictions on our cocycle, instead, we have to leave out those which are coboundaries and could thus be compensated by higher Massey products.

Let ω be a cocycle given by its coefficients $a_{i,j}$. Note that we can still suppose $a_{1,j} = 0$ for all $j \geq -l + 1$ and all $j \leq -l - 1$ (cf. 2.0, 2.3, 2.4). On the other hand, we may suppose that we are in the stable range and by 2.4 that the first terms in the 3 column (i.e. at least $a_{3,4}$, $a_{3,5}$) are zero up to a coboundary. As we cannot assume simultaneously that $a_{3,4}$, $a_{3,5}$ and $a_{1,-l} = a_{1,2}$ are zero (cf. 2.4), we choose to allow $a_{1,2}$ non-zero.

When writing a Massey square M_{ijk} , we will now always suppose that the indices are ordered $i < j < k$. The Massey squares which may be compensated are those M_{ijk} with $i + j + 2l = 1$, according to Section 3.1. This means in weight $l = -2$ that all M_{23k} can be compensated, and that these are the only (ordered) ones. The other M_{ijk} must be zero.

We start now a case study in order to determine which possibilities there are for ω , imposing that all (ordered) Massey squares M_{ijk} with $i + j \neq 5$ are zero.

1st case. Suppose $a_{2,3} = a_{2,4} = 0$. Then $a_{2,5} = -a_{3,4} = -a_{3,5}$ and $a_{2,5} = a_{3,5} + a_{2,6}$ implying that $a_{2,6} = -2a_{3,5}$. $M_{245} = a_{4,5}(a_{2,5} - a_{2,7}) = 0$.

Case 1a. $a_{2,5} = 0$, and thus $a_{3,4} = a_{3,5} = 0$, $a_{2,6} = 0$. Then either $a_{4,5} = 0$ ($\Rightarrow a_{3,6} = 0$, $a_{2,7} = 0$), or $a_{2,7} = 0$ ($\Rightarrow a_{3,6} = 0$, $a_{4,5} = 0$). In any case $a_{2,7} = 0$, $a_{3,6} = 0$ and $a_{4,5} = 0$.

Suppose now given r ($r \geq 10$) such that $a_{i,j} = 0$ for all $i + j \leq r$. Then $M_{2ij} = a_{i,j}(a_{2,i} - a_{2,i+j-2} + a_{2,j})$ must be zero for $i \geq 4$. Let us suppose $i < j$ (indices ordered!) and $i + j = r + 1$. Then by hypothesis $a_{2,i} = a_{2,j} = 0$ and $M_{2jk} = -a_{i,j}a_{2,r-1}$. Thus either $a_{i,j} = 0$, or $a_{2,r-1} = 0$. But these two elements are on a new diagonal (in the matrix of coefficients $a_{i,j}$), and all elements with lower indices are zero. By Eq. (2) this implies that two (because approaching the diagonal, one jumps to the next diagonal by the diagonal equations (cf. 2.5) $a_{s,s+1} = a_{s,s+2}$) new diagonals are zero, and by induction, all coefficients are zero in this case.

Case 1b. $a_{2,5} = 1$, and thus $a_{2,6} = 2$, $a_{3,4} = a_{3,5} = -1$. Now $M_{245} = 0$ implies $a_{4,5} = 0$ or $a_{2,7} = 1$. But for $a_{2,7} = 1$, we get by repeated use of Eq. (2) $a_{3,6} = 1$, $a_{4,5} = -2$, $a_{4,6} = -2$,

$a_{3,7} = 3$ and finally $a_{2,8} = -2$. Relate this then to $M_{246} = a_{4,6}(-a_{2,8} + a_{2,6}) \neq 0$, to conclude that $a_{2,7} \neq 1$, and therefore $a_{4,5} = 0$.

This means that all $a_{i,j}$ with $i + j \leq 10$ are the same as for the 3-family (cf. 2.5). Let us show the induction step in order to conclude that two more diagonals are like in the 3-family. Indeed, suppose now $a_{i,j}$ with $i + j \leq r$ like in the 3-family, and take $j = r - 3$. We have $M_{24j} = a_{4,j}(a_{2,j} - a_{2,j+2}) = a_{4,r-3}(a_{2,r-3} - a_{2,r-1}) = 0$. The coefficient $a_{2,r-3}$ must be as in the 3-family by hypothesis. We want to conclude that $a_{4,r-3} = 0$ (as in the 3-family), opening up two more diagonals. Therefore we show that $a_{2,r-3} \neq a_{2,r-1}$.

Let us denote $a_{2,r-3} = t$. We have by hypothesis $a_{2,r-2} = t + 1$, $a_{3,r-4} = a_{3,r-3} = -1$, and $a_{k,s} = 0$ for $k < s$, $k \geq 4$. Suppose $a_{2,r-3} = a_{2,r-1}$, and this will lead to $a_{4,r-2} \neq 0$ while $a_{2,r-2} = t + 1$ and $a_{2,r} \neq t + 1$. More precisely, in the new diagonal starting from $a_{2,r-1} = t$, we get $a_{3,r-2} = 1$, $a_{4,r-3} = -2$, $a_{5,r-4} = 2$, and then we always get ± 2 , because there are only zeroes one diagonal higher. By construction $r - 1$ is odd, say $r - 1 = 2k + 1$. Doing in this sense $k - 1$ steps on the diagonal towards the diagonal transforms $a_{2,r-1} = a_{2,2k+1}$ into $a_{k-1+2,k+2} = a_{k+1,k+2}$. But by the diagonal equation (cf. 2.5), $a_{k+1,k+2} = a_{k+1,k+3}$, and then we work back $k - 3$ steps to get $\pm 2(k - 3)$, and finally $a_{2,r} = -(\mp 2(k - 3) + 1) + t$. This is equal to $t + 1$ only if $k = 2$ or $k = 4$. $k = 2$ is already treated, and for $k = 4$, one can check directly that $a_{2,8} \neq a_{2,10}$:

	2	3	4	5
0				
0	-1			
1	-1	0		
2	-1	0	2	
3	-1	-2	2	
4	1	-4		
3	5			
-8				

In conclusion, the only non-zero cocycle (making zero the non-compensable Massey squares) compatible with case 1 is the 3-family.

2nd case. Here we can take $a_{2,3} = a_{2,4} = 1$. Recall that we choose to take the first terms in the 3 column (i.e. at least $a_{3,4}$, $a_{3,5}$) to be zero (possibly by adding a coboundary).

Case 2a. Suppose as a first subcase $a_{4,5} = a_{4,6} = 0$. Then we have up to $a_{i,j}$ with $i + j = 10$ the 2-family. Set $a_{5,6} = a$. We get then $a_{2,9} = 1 - a$ and $a_{2,10} = 1 - 4a$ by repeated use of Eq. (2). But $M_{247} = a_{4,7}(a_{2,4} - a_{2,9} + a_{2,7}) = 0$ implies $a = 0$ or $a = -1$, while $M_{248} = a_{4,8}(a_{2,4} - a_{2,10} + a_{2,8}) = 0$ implies $a = 0$ or $a = -\frac{1}{4}$. In conclusion, $a = 0$ and the 2-family is reproduced one diagonal higher. Using M_{24j} and $M_{24(j+1)}$, one can show in a similar way that the only solution here is the 2-family.

Case 2b. Here $a_{2,3} = a_{2,4} = 1$, $a_{3,4} = a_{3,5} = 0$, but $a := a_{4,5} = a_{4,6} \neq 0$. By $M_{245} = a_{4,5}(a_{2,4} - a_{2,7} + a_{2,5})$, $M_{246} = a_{4,6}(a_{2,4} - a_{2,8} + a_{2,6})$ and $M_{345} = a_{4,5}(a_{3,5} - a_{3,7})$, we get thus $a_{3,5} = a_{3,6} = 0$, $a_{2,4} + a_{2,5} = a_{2,7}$ and $a_{2,4} + a_{2,6} = a_{2,8}$. But then, on the one hand, $a_{2,3} = a_{2,4} = a_{2,5} = a_{2,6} = a_{2,7}$ by Eq. (2), but, on the other hand, $a_{2,4} + a_{2,5} = a_{2,7} = 2$, which is a contradiction.

As a conclusion of the case study, the only cocycles which can possibly give true deformations are the 2- and the 3-family, but possibly with a non-zero $a_{1,2}$ coefficient.

The 2-family (cf. 2.5) is a square zero 2-cocycle in weight $l = -2$, it is not contradictory in weight -2 (cf. 2.7), and is thus one solution here.

The 3-family ω has a non-zero Massey square, namely $M_{23j} = a_{2,j} - a_{2,j+1} = 1$ for all $j \geq 4$. Let us show that the corresponding Massey cube is then zero, and thus that the 3-family gives indeed rise to a true deformation in weight -2 :

We must write M_{23k} as a coboundary. The cochain $\alpha(e_i, e_j) = b_{i,j}e_{i+j-4}$ with $b_{i,j} = M_{23k}$ for $i = 2$ and $j = 3$, $b_{i,j} = M_{32k}$ for $i = 3$ and $j = 2$, and $b_{i,j} = 0$ otherwise satisfies

$$d\alpha(e_i, e_j, e_k) = M_{23k}e_{k+1}.$$

We must then compute the Massey cube

$$N_{ijk} := \alpha(\omega(e_i, e_j), e_k) + \omega(\alpha(e_i, e_j), e_k) + \text{cycl.} = a_{i,j}b_{i+j-2,k} + b_{i,j}a_{i+j-2,k} + \text{cycl.}$$

But if $b_{i+j-2,k} \neq 0$, then $i + j - 2, k \in \{2, 3\}$. The only possibly non-zero term is thus $N_{23k} = M_{23k}a_{1,k} = 0$ ($k \geq 4$ here).

Finally, let us show that we cannot get any information about $a_{1,2}$, neither by the cocycle equations, nor by the vanishing of the Massey squares. This is clear for the cocycle equations. Let us show that we cannot deduce $a_{1,2} = 0$ from Massey squares which have to vanish. Indeed, when writing down the Massey squares which involve $a_{1,2}$, the only possibly non-zero Massey squares M_{ijk} involving $a_{1,2}$ (with ordered indices) have $i = 1$. But then we have to have $j = 2$ in order to involve $a_{1,2}$. One easily checks that $M_{12k} = 0$.

To summarize, we have the following

Proposition 3. *The 3-family in weight -2 has a non-zero Massey square, but its Massey cube is zero, and we get consequently a true deformation. In weight -2 , the 2- and 3-family with possibly a non-zero term $a_{1,2}$ define the only cohomology classes leading to true deformations.*

4.3. True deformations in weight -3

We will determine all cocycles leading to true deformations in weight -3 once again by imposing on a general cocycle ω given by its coefficients $a_{i,j}$ for all $i, j \geq 2$, $i \neq j$, that all Massey squares which cannot possibly be compensated (cf. Section 3.1) are zero. The squares which cannot serve to give conditions on the $a_{i,j}$ are those M_{ijk} (with ordered indices $i < j < k$) with $i + j = 7$, $i + k = 7$ or $j + k = 7$.

All 1-coefficients other than $a_{1,2}$ and $a_{1,3}$ may be supposed to be zero by 2.3 (cf. 2.4). $a_{1,2} = 0$ by 2.0. We choose once again that the first 4-coefficients (i.e. at least $a_{4,5} = a_{4,6}$) are zero, up to a coboundary, according to 2.5, while not imposing anything on $a_{1,3}$ (cf. 2.4).

Let us draw the table for the coefficients of ω :

2	3	4	5
a			
a	b		
$a - b$	b	0	
$a - 2b$	b	0	
$a - 3b$	b		
$a - 4b$			

Now we write down the Massey squares that we may use: $M_{236} = a_{2,6}(a_{2,3} - a_{3,6} + a_{3,5}) = a(a - 2b)$, $M_{237} = a_{2,7}(a_{2,3} - a_{3,7} + a_{3,6}) = a(a - 3b)$, $M_{246} = a_{2,4}a_{3,6} + a_{4,6}a_{7,2} + a_{6,2}a_{5,4} = ab$.

In conclusion, $a = 0$. But then up to $i + j = 10$, the 3-family has built up. Let us show by induction that the 3-family is the only possible solution:

Suppose the $a_{i,j}$ up to $i + j = r$ for $r \geq 10$ are like in the 3-family (cf. 2.5). Consider the Massey square

$$M_{23k} = a_{2,3}a_{2,k} + a_{3,k}a_{k,2} + a_{k,2}a_{k-1,3} = a_{2,k}(a_{2,3} - a_{3,k} + a_{3,k-1}).$$

We may use its vanishing to deduce restrictions on the $a_{i,j}$ as soon as $k \geq 6$. For $k \geq 8$ and with $r = k + 2$, $M_{23k} = 0$ implies under the induction hypothesis that $a_{3,k} = a_{3,k-1}$, and we have therefore transmitted the 3-family to two more diagonals, showing the induction step.

In order to conclude, let us show that the 3-family is of Massey square zero:

Recall that the 3-family is defined by $a_{3,k} = a \neq 0$, $a_{j,k} = 0$ for all $j, k \geq 4$, and $a_{2,3} = a_{2,4} = 0$, $a_{2,k} = -(k - 4)a$ for all $k \geq 5$.

It is clear that $M(a)_{ijk} = 0$ for all $i, j, k \geq 4$, by definition of the 3-family. Suppose $i = 3$ ($j = 3$ or $k = 3$ would be a symmetric case):

$$M(a)_{3jk} = a_{3,j}a_{j,k} + a_{j,k}a_{j+k-3,3} + a_{k,3}a_{k,j} = a_{j,k}(a_{3,j} + a_{j+k-3,3} + a_{3,k}).$$

This last expression is zero if both j and k are greater or equal to 4 (as then $a_{j,k} = 0$), and also if one of them is equal to 2, because in this case the term in parenthesis is zero. Suppose now that $i = 2$,

$$M(a)_{2jk} = a_{2,j}a_{j-1,k} + a_{j,k}a_{j+k-3,2} + a_{k,2}a_{k-1,j}.$$

In case $j, k \geq 4$, this expression reduces to $a_{2,j}a_{j-1,k} + a_{k,2}a_{k-1,j}$ which is evidently zero if both j and k are greater or equal to 5, and in case $j = 4$, $a_{k-1,j}$ and $a_{2,j}$ are zero. It remains the case where $j = 3$, but then we get $a_{3,k}a_{k,2} + a_{k,2}a_{k-1,3} = 0$.

Finally, let us show that the possibly non-zero coefficient $a_{1,3}$ cannot be shown to be zero using the vanishing of Massey squares. The only M_{ijk} (with ordered indices) involving $a_{1,3}$ have $i = 1$,

$$M_{1jk} = a_{1,j}a_{j+1+l,k} + a_{j,k}a_{j+k+l,1} + a_{k,1}a_{k+1+l,j}.$$

Then for $j = 3$, we get $M_{13k} = 0$, and for $j + k = 6$, we get also $M_{1jk} = 0$, and these are the only combinations (up to reordering) involving $a_{1,3}$.

To summarize, we get the following

Proposition 4. *In weight -3 , the 3-family, with a possibly non-zero $a_{1,3}$ coefficient, defines the only cohomology class leading to an true deformation.*

4.4. True deformations in weight -4

We will determine all cocycles leading to true deformations in weight -4 once again by imposing on a general cocycle ω given by its coefficients $a_{i,j}$ for all $i, j \geq 2$, $i \neq j$ that all Massey squares which cannot possibly be compensated (cf. Section 3.1) are zero. The squares which cannot serve to give conditions on the $a_{i,j}$ are those M_{ijk} (with ordered indices $i < j < k$) with $i + j = 9$, $i + k = 9$ or $j + k = 9$. In weight $l = -4$, we have to be more careful with the conditions as we are not always in the stable range (cf. 2.5). For example, 2.1 implies here that $a_{2,4} = 0$ (and we cannot deduce here $a_{2,3} = a_{2,4}$). But for $j > i \geq 3$, and for $i = 2$ and $j \geq 4$ we still have

$$a_{i+1,j} + a_{i,j+1} = a_{i,j}.$$

But then 2.2 implies that $a_{2,3} = 0$. All 1-coefficients other than $a_{1,2}$, $a_{1,3}$ and $a_{1,4}$ may be supposed to be zero by 2.3, while $a_{1,2} = a_{1,3} = 0$ follows from 2.0. We choose once again according to 2.4 that the first 5-coefficients (i.e. at least $a_{5,6} = a_{5,7}$) are zero, up to a coboundary, while we do not impose anything on $a_{1,4}$, cf. 2.4.

Let us draw the table for the coefficients of ω :

2	3	4	5
0			
0	b		
$-b$	b	c	
$-2b$	$b - c$	c	0
$c - 3b$	$b - 2c$	c	0
$3c - 4b$	$b - 3c$	c	
$6c - 5b$	$b - 4c$		
$10c - 6b$			

Now we write down the Massey squares that we may use: $M_{246} = a_{4,6}a_{6,2} = 2cb$, $M_{238} = a_{3,8}a_{7,2} + a_{8,2}a_{6,3} = (b - 3c)(3b - c) + (3c - 4b)(b - c) = -b(b + 3c)$, $M_{256} = a_{2,5}a_{3,6} + a_{5,6}a_{7,2} + a_{6,2}a_{4,5} = b(-b + 3c)$.

Now start a case study: either $b = 0$, and in this case we want to show that the 4-family is built up by induction. Indeed, we have $M_{24k} = a_{2,k}(a_{4,k-2} - a_{4,k})$ which must vanish as soon as $k \geq 8$. In this way we transmit the built up 4-family to another two diagonals. Or $c = 0$ and in this case all coefficients are zero. The zero family is also easily shown to be built up from this initial stage.

Let us show that the nullity of $a_{1,4}$ cannot be derived from the nullity of Massey squares. The Massey squares (with ordered indices) where $a_{1,4}$ shows up, have either $i = 1$ or they are M_{234} . The latter is zero anyhow, and the former are shown to be zero as for $l = -1$, -2 and -3 .

To summarize, we have the following

Proposition 5. *The only cohomology class leading to an true deformation in weight $l = -4$ is represented by the 4-family, with a possibly non-zero coefficient $a_{1,4}$.*

The fact that the 4-family has zero Massey square in weight -4 follows from Proposition 7 in Section 4.5.

4.5. True deformations in weight l , $l \leq -5$

We will show that in degree l , the $-l$ family is of Massey square zero, and that this is the only family for which all Massey squares which cannot be compensated, are zero. Therefore, we will show that in weight l , $l \leq -5$, the $-l =: m$ family, with a possibly non-zero coefficient $a_{1,m}$, is the only cocycle which leads to true deformations.

For this, we need the explicit expression of the non-zero low degree coefficients of the $-l$ family. It is obvious from 2.5 how to deduce the expressions of the coefficients of the general $m := -l$ family from those for the low degree families:

The m family. $a_{m,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq m + 1$, $a_{m-1,m} = a_{m-1,m+1} = 0$, $a_{m-1,k} = -(k - (m + 1))$ for all $k \geq m + 2$, $a_{m-2,m-1} = a_{m-2,m} = a_{m-2,m+1} = a_{m-2,m+2} = 0$, $a_{m-2,k} = \frac{(k-(m+1))(k-(m+2))}{2}$ for all $k \geq m + 3$, and $a_{m-3,m-2} = a_{m-3,m-1} = a_{m-3,m} = a_{m-3,m+1} = a_{m-3,m+2} = a_{m-3,m+3} = 0$, $a_{m-3,k} = -\frac{(k-(m+1))(k-(m+2))(k-(m+3))}{3!}$ for all $k \geq m + 4$, and so on.

Proposition 6. *The m -family defines a 2-cocycle in any weight.*

Proof. We have to show that the m family satisfies the requirements of Sections 2.1 (i.e. Eq. (2); observe that with the non-zero coefficients of the m -family, we are always in the stable range) and 2.2. It is clear that the requirement of 2.2 is met.

For Eq. (2), take the general expression of the above coefficients

$$a_{m-r,k} = \pm \frac{(k - (m + 1))!}{r!(k - (m + r + 1))!}$$

for all $k \geq m + r + 1$, and all $r \leq m - 2$; \pm denotes an alternating sign with respect to the parity of r . Now

$$a_{m-r,k+1} + a_{m-(r-1),k} = \frac{(k+1-(m+1)-r)(k-(m+1))!}{r!(k-(m+r))!} = a_{m-r,k}. \quad \square$$

Let us show now that the Massey square of the m -family is zero (i.e. not only the non-compensable Massey squares, but all).

Proposition 7. *All Massey squares of the m -family are zero in weight $l = -m$.*

Proof. Indeed, we have

$$M_{ijk} = a_{i,j}a_{i+j+l,k} + a_{j,k}a_{j+k+l,i} + a_{k,i}a_{k+i+l,j}.$$

We will always consider ordered Massey squares, i.e. M_{ijk} with $i < j < k$, and it will be enough to show that these are zero.

First case. $i = m - r$, $j = m - p$, and $k \geq m + 1$ with $p, r \geq 0$. These conditions imply $a_{i,j} = 0$, and we get

$$\begin{aligned} M_{ijk} &= a_{j,k}a_{j+k+l,i} + a_{k,i}a_{k+i+l,j} \\ &= (-1)^{p+1} \frac{(k - (m + 1))!}{p!(k - (m + p + 1))!} (-1)^r \frac{(k - p - (m + 1))!}{r!(k - p - (m + p + 1))!} \\ &\quad + (-1)^{r+1} \frac{(k - (m + 1))!}{r!(k - (m + r + 1))!} (-1)^{p+1} \frac{(k - r - (m + 1))!}{p!(k - r - (m + p + 1))!}. \end{aligned}$$

Suppose now first that $k + i + l = k - r > j$. In this case we get by taking out common factors

$$M_{ijk} = (-1)^{p+r+1} \frac{(k - (m + 1))!}{r!p!(k - p - (m + r + 1))!} \left(\frac{(k - p - (m + 1))!}{(k - (m + p + 1))!} - \frac{(k - r - (m + 1))!}{(k - (m + r + 1))!} \right)$$

which is zero. On the other hand, in case $k + i + l = k - r = j$, we get $k = m + q$, $i = m - r$, $j = m - p$. Then the only possibly non-zero term is $M_{ijk} = a_{j,k}a_{j+k+l,i}$, because $a_{k+i+l,j} = 0$. But $a_{j,k}a_{j+k+l,i} = a_{m-p,m+q}a_{m+(q-p),m-(p+q)}$ and $a_{m+(q-p),m-(p+q)} = 0$, because $a_{m-s,m+s} = 0$, $a_{m-s,m+s+1} \neq 0$ marks the last zero term in the m family (when fixing $m - s$ and counting up the second index), but here $q - p \leq p + q$. It remains the third subcase where $k - r < j$, but then $r > p + q$, and thus $a_{j+k+l,i} = a_{m+q-p,m-r} = 0$ and $a_{k,i} = a_{m+q,m-r} = 0$ by the same reasoning as before. So the first case is settled.

Second case. $i = m - r$, $j = m + p$, and $k = m + q$ still with $i < j < k$, i.e. $q > p$. These conditions imply $a_{j,k} = 0$, and we get

$$\begin{aligned} M_{ijk} &= a_{i,j}a_{i+j+l,k} + a_{k,i}a_{k+i+l,j} \\ &= a_{m-r,m+p}a_{m-(r-p),m+q} - a_{m-r,m+q}a_{m-(r-q),m+p}. \end{aligned}$$

Now we study the relative position of r to q : if first $r \geq q$, then $a_{m-r,m+q} = 0$ and $a_{m-r,m+p} = 0$. If $r < q$, then $a_{m-(r-q),m+p} = 0$ and following the relative position of r to p , either $a_{m-(r-p),m+q} = 0$ ($r < p$) or $a_{m-r,m+p} = 0$ ($r \geq p$). In any case, all terms are zero. \square

We now come to the last and main point of this section, namely the proof that the m family is the only family in weight $l = -m \leq -5$ which satisfies the vanishing of all Massey squares which cannot be compensated, i.e. of all Massey squares whose vanishing is necessary in order to have an true deformation.

Let therefore ω be a cocycle given by its coefficients $a_{i,j}$. By 2.1, we have for $i + j \geq m + 2$ the usual (or stable) cocycle identity $a_{i+1,j} + a_{i,j+1} = a_{i,j}$, and for $i + j = m, m + 1$ just $a_{i+1,j} + a_{i,j+1} = 0$ while there is no equation for lower $i + j$. By 2.2, we have $a_{i,j} = 0$ for $i + j = m + 1$, compatible with the foregoing statements. By 2.0, the coefficients $a_{1,2}, \dots, a_{1,m-1}$ are zero, while by 2.3 $a_{1,m+1}, a_{1,m+2}, \dots$ may be taken to be zero. Once again, we do not impose anything on $a_{1,m}$ in order to use the freedom of choice for a coboundary to take the first coefficients (from the diagonal) in the $m + 1$ st column to zero, according to 2.4.

The Massey squares which can be compensated and thus do not impose conditions on ω are the M_{ijk} with $i + j = 2m + 1$, $j + k = 2m + 1$ or $k + i = 2m + 1$.

Let us draw a diagram of the coefficients of ω :

	$m - 2$	$m - 1$	m	$m + 1$	$m + 2$
$m - 1$	a				
m	a	b			
$m + 1$	$a - b$	b	c		
$m + 2$	$a - 2b$	$b - c$	c	0	
$m + 3$	$a - 3b + c$	$b - 2c$	c	0	e
$m + 4$	$a - 4b + 3c$	$b - 3c$	c	$-e$	e
$m + 5$	$a - 5b + 6c$	$b - 4c$	$c + e$	$-2e$	
$m + 6$	$a - 6b + 10c$	$b - 5c - e$	$c + 3e$		
$m + 7$	$a - 7b + 15c + e$	$b - 6c - 4e$			
$m + 8$	$a - 8b + 21c + 5e$				

Let us also expose some Massey squares M_{ijk} (such that no sum of pairs of indices gives $2m + 1$):

$$M_{m-1,m,m+3} = a_{m-1,m}a_{m-1,m+3} + a_{m,m+3}a_{m+3,m-1} + a_{m+3,m-1}a_{m+2,m} = b(b - 2c),$$

$$M_{m-1,m+1,m+3} = a_{m-1,m+1}a_{m,m+3} + a_{m+1,m+3}a_{m+4,m-1} + a_{m+3,m-1}a_{m+2,m+1} = bc,$$

$$M_{m,m+2,m+3} = a_{m,m+2}a_{m+2,m+3} + a_{m+2,m+3}a_{m+5,m} + a_{m+3,m}a_{m+3,m+1} = e(c - e),$$

$$M_{m-2,m,m+2} = a_{m-2,m}a_{m-2,m+2} + a_{m,m+2}a_{m+2,m-2} = (a - c)(a - 2b),$$

$$M_{m-2,m,m+4} = a_{m-2,m}a_{m-2,m+4} + a_{m,m+4}a_{m+4,m-2} + a_{m+4,m-2}a_{m+2,m} = a(a - 4b + 3c),$$

$$\begin{aligned} M_{m-2,m,m+5} &= a_{m-2,m}a_{m-2,m+5} + a_{m,m+5}a_{m+5,m-2} + a_{m+5,m-2}a_{m+3,m} \\ &= (a - 5b + 6c)(a - e). \end{aligned}$$

$M_{m-1,m+1,m+3} = bc = 0$. Now start a case study:

First case. $b = 0$, then by $M_{m-2,m,m+2} = 0$, either $a = 0$ or $a = c$. In the first subcase, $M_{m-2,m,m+5} = 0$ implies $ce = 0$, thus the only possibly non-zero parameter is c by $M_{m,m+2,m+3} = 0$. Note that a non-zero c corresponds to the m family. In the second subcase, $a = c$ and then $M_{m-2,m,m+4} = 0$ implies $a = 0$. Finally $a = b = c = e = 0$ by $M_{m,m+2,m+3} = 0$.

Second case. $c = 0$, then by $M_{m,m+2,m+3} = 0$, $e = 0$, by $M_{m-1,m,m+3} = 0$, $b = 0$, and finally by $M_{m-2,m,m+4} = 0$, $a = 0$.

Now it is clear how to perform an induction step showing that the m family is transmitted to a next two diagonals for example using $M_{m-2,m,m+7}$:

$$M_{m-2,m,m+7} = a_{m-2,m+7}(a_{m-2,m} - a_{m,m+7} + a_{m,m+5}),$$

and by assumption $a_{m-2,m+7} \neq 0$, $a_{m-2,m} = 0$, and thus by $M_{m-2,m,m+7} = 0$, $a_{m,m+7} = a_{m,m+5} = c$ which is the induction step. This shows that starting from the $(m-2)$ nd column, all coefficients are as in the m family. In order to come to lower coefficients, take for example

$$M_{m-3,m+1,m+5} = a_{m-3,m+1}a_{m-2,m+5} + a_{m+1,m+5}a_{m+6,m-3} + a_{m+5,m-3}a_{m+2,m+1}.$$

Here, $a_{m+1,m+5} = a_{m+2,m+1} = 0$ and $a_{m-2,m+5} \neq 0$ by assumption, therefore $a_{m-3,m+1} = 0$ which transmits the m -family to the $(m-3)$ rd column.

Finally, let us argue that the coefficient $a_{1,m}$ cannot be shown to be zero by the vanishing of Massey squares. Indeed, in order to involve $a_{1,m}$, the Massey square M_{ijk} (with ordered indices) must have either $i = 1$ or $i + j + l = 1$, $j + k + l = 1$ or $i + k + l = 1$. The first alternative is rather easily seen to be zero. Fix $i + j + l = 1$ for the second alternative. It describes a situation where the coefficient $a_{1,m}$ is multiplied by $a_{i,j}$ with $i + j = m + 1$. This coefficient is zero.

To summarize, we have the following

Proposition 8. *The only non-zero cohomology class compatible with the vanishing of all Massey squares which cannot be compensated, is the m -family in weight $l = -m \leq -5$, with a possibly non-zero coefficient $a_{1,m}$.*

5. Deformations in zero and positive weights

5.1. True deformations in weight $l = 0$

In weight $l \geq 0$, a new phenomenon is happening: we have a relation between the Massey squares. Recall that the cocycle coefficients $a_{i,j}$ are supposed to be antisymmetric in i, j , and that $a_{i,i}$ is set to zero for all i .

Proposition 9. *Let i, j, k , be three integers, $i, k \geq 2$ and $j \geq 3$. We have the relation*

$$M_{ijk} + M_{i(j-1)(k+1)} + M_{(i+1)(j-1)k} = M_{i(j-1)k}.$$

Proof. We have by definition

$$\begin{aligned} M_{ijk} + M_{i(j-1)(k+1)} &= a_{i,j}a_{i+j+l,k} + a_{j,k}a_{j+k+l,i} + a_{k,i}a_{i+k+l,j} \\ &\quad + a_{i,j-1}a_{i+j-1+l,k+1} + a_{j-1,k+1}a_{j+k+l,i} + a_{k+1,i}a_{i+k+1+l,j-1}. \end{aligned}$$

We transform the terms $a_{j,k}a_{j+k+l,i} + a_{j-1,k+1}a_{j+k+l,i}$, using repeatedly the cocycle Eq. (2) to

$$a_{j+k+l,i}(a_{j,k} + a_{j-1,k+1}) = a_{j+k+l,i}a_{j-1,k} = a_{j-1,k}(a_{j+k-1+l,i} - a_{j+k-1+l,i+1}).$$

We transform the terms $a_{i,j}a_{i+j+l,k} + a_{i,j-1}a_{i+j-1+l,k+1}$, using Eq. (2) to

$$a_{i,j}a_{i+j+l,k} + a_{i,j-1}a_{i+j-1+l,k} - a_{i,j-1}a_{i+j+l,k}.$$

We transform the terms $a_{k,i}a_{i+k+l,j} + a_{k+1,i}a_{i+k+1+l,j-1}$, using Eq. (2) to

$$a_{k,i}a_{i+k+l,j-1} - a_{k,i}a_{i+k+1+l,j-1} + a_{k+1,i}a_{i+k+1+l,j-1}.$$

In these three transformations, the sum of the first term of the first, the second term of the second and the first term of the third give together

$$a_{j-1,k}a_{j+k-1+l,i} + a_{i,j-1}a_{i+j-1+l,k} + a_{k,i}a_{i+k+l,j-1} = M_{i(j-1)k}.$$

The remaining terms read

$$-a_{j-1,k}a_{j+k-1+l,i+1} + a_{i,j}a_{i+j+l,k} - a_{i,j-1}a_{i+j+l,k} - a_{k,i}a_{i+k+1+l,j-1} + a_{k+1,i}a_{i+k+1+l,j-1}.$$

Here, the second and third term give

$$a_{i,j}a_{i+j+l,k} - (a_{i,j}a_{i+j+l,k} + a_{i+1,j-1}a_{i+j+l,k}) = -a_{i+j+l,k}a_{i+1,j-1},$$

while the last two terms give

$$-(a_{k+1,i} + a_{k,i+1})a_{i+k+1+l,j-1} + a_{k+1,i}a_{i+k+1+l,j-1} = -a_{i+k+1+l,j-1}a_{k,i+1},$$

still using Eq. (2).

In summary, the remaining terms give

$$-a_{j-1,k}a_{j+k-1+l,i+1} - a_{i+j+l,k}a_{i+1,j-1} - a_{i+k+1+l,j-1}a_{k,i+1} = -M_{(i+1)(j-1)k}.$$

This ends the proof of the lemma. \square

Corollary 1.

$$M_{i(i+1)k} + M_{i(i+2)(k-1)} = M_{i(i+1)(k-1)}.$$

Observe that also repeated indices may give interesting relations: for example, for $i = 2$, $j = 4$ and $k = 4$, we get $M_{234} = M_{235}$. It is easily shown by these relations that the nullity of M_{23k} for all $k \geq 4$ is necessary for the nullity of all Massey squares, and that the nullity of M_{2rs} for all $r, s \geq 3$ is necessary and sufficient for the nullity of all Massey squares. We believe that the minimal set of Massey squares whose nullity implies the nullity of all Massey squares is somewhere in between these two sets, but we could not get hold on it.

Now, we will determine all square zero cocycles, i.e. all true deformations of m_0 , in weight $l = 0$: first of all, the 2-family is such a cocycle. Then, let us suppose that ω is a non-trivial 2-cocycle which is independent of the 2-family and has zero Massey squares; as before, we think of ω as given by the coefficients $a_{i,j}$, and we will distribute letters to its initial terms: $a_{2,3} = a$, $a_{3,4} = b$, and so on.

Using Eq. (2), we establish the following diagram which is of course valid for all weights l ; observe that the general expression for the coefficients in Section 4.5, proof of Proposition 6,

leads for general coefficients $a_{2,3} =: u_2 = a$, $a_{3,4} =: u_3 = b$, $a_{4,5} =: u_4 = c$ and so on (by linearity) to the formula

$$a_{i,j} = \sum_{m=2}^{j-1} (-1)^{m-i} u_m \frac{(j - (m+1))!}{(m-i)!(j-2m+i-1)!}, \quad (4)$$

which may be used to compute the coefficients in the following diagram more easily (than by a recursive formula).

	2	3	4	5	6	7
3	a					
4	a	b				
5	$a - b$	b	c			
6	$a - 2b$	$b - c$	c	d		
7	$a - 3b + c$	$b - 2c$	$c - d$	d	e	
8	$a - 4b + 3c$	$b - 3c + d$	$c - 2d$	$d - e$	e	f
9	$a - 5b + 6c - d$	$b - 4c + 3d$	$c - 3d + e$	$d - 2e$	$e - f$	f
10	$a - 6b + 10c - 4d$	$b - 5c + 6d - e$	$c - 4d + 3e$	$d - 3e + f$	$e - 2f$	
11	$a - 7b + 15c - 10d + e$	$b - 6c + 10d - 4e$	$c - 5d + 6e - f$	$d - 4e + 3f$		
12	$a - 8b + 21c - 20d + 5e$	$b - 7c + 15d - 10e + f$	$c - 6d + 10e - 4f$			
13	$a - 9b + 28c - 35d + 15e - f$	$b - 8c + 21d - 20e + 5f$				
14	$a - 10b + 36c - 56d + 35e - 6f$					

From now on, we consider weight $l = 0$.

Order the Massey squares by their *level*, i.e. we say that M_{ijk} has level $i + j + k$. Computing Massey squares and setting them equal to zero gives an infinite family of homogeneous quadratic equations for the infinite family of variables a, b, c, d, \dots

In level 9, the only Massey square is M_{234} , and its nullity gives

$$3b^2 - bc - 2ac = 0.$$

In level 10, the only Massey square is M_{235} , and its nullity gives the same equation. In level 11, there are Massey squares M_{236} and M_{245} , and their nullity gives (possibly by subtracting the previous equation) in both cases

$$2ad - 4bc - bd + 6c^2 - cd = 0.$$

In level 12, there are Massey squares M_{237} , M_{246} and M_{345} , and their nullity gives (possibly by subtracting the previous equations) in all cases

$$-3bd + 4c^2 - 3cd = 0.$$

Going higher in this hierarchy of equations and variables, there are at each new level some (possibly) linear independent equation. Proposition 9 only tells us that the nullity of M_{2rs} with $2 < r < s$ is enough in order to have all Massey squares zero. We do not know which of these equations are in fact the independent one's.

In Massey square level 14, we arrive at 5 equations for the 5 variables a, b, c, d, e , which read (after subtracting t each step multiples of the previous equations):

$$\begin{aligned}
3b^2 - bc - 2ac &= 0, \\
2ad - 4bc - bd + 6c^2 - cd &= 0, \\
-3bd + 4c^2 - 3cd &= 0, \\
e(-2a + 3b - d) + 5bd - 15cd + 10d^2 &= 0, \\
e(-6a + 15b - 4c - 11d) - 55cd + 50d^2 + 15bd &= 0.
\end{aligned}$$

The discussion of these equations (either by hand or by a system computing a Gröbner basis for the homogeneous polynomials) gives as non-zero solutions the 2-family and one other family with coefficients $a = \frac{1}{6}$, $b = \frac{1}{60}$, $c = \frac{1}{420}$, etc. We describe this family from another point of view in Section 5.3, which will show that this family must verify all equations and not only the five equations we wrote down. These are the only square zero solutions in weight 0, and we have determined all true deformations in this case.

5.2. True deformations in weight $l > 0$

In the weight $l = 1$ case, we get from the same diagram as in weight $l = 0$ up to Massey level 15 (where we took only the equations of type M_{23k} in order to simplify) six homogeneous quadratic equations in six variables which read:

$$\begin{aligned}
-3ac + 4b^2 - 3bc &= 0, \\
-5bc - 2bd + 10c^2 - 4cd + 3ad &= 0, \\
5c^2 - 4bd + 2be + ec - 6cd &= 0, \\
e(-3a + 11b - 5d + 3c) - 6bd + 15c^2 - 39cd + 20d^2 &= 0, \\
e(-9a + 35b - 35d) + f(-4b + 2c + 2d) - 6bd + 30c^2 - 111cd + 90d^2 &= 0, \\
e(-18a + 75b + 11c - 186d + 35e - 6f) + f(3a + 20c - 24b + 16d) \\
- 4bd + 50c^2 - 234cd + 252d^2 &= 0.
\end{aligned}$$

By a computation with MUPaD which determines a Gröbner basis for the homogeneous polynomials corresponding to these equations (actually we took here all equations of type M_{2rs}), one obtains as (non-zero) solutions the 2-family, a solution $a = 0$, $b = 0$, $c = 0$, $d = 0$ and $e = 1$, $f = \frac{35}{6}$, and a further solution $a = 1$, $b = \frac{1}{7}$, $c = \frac{1}{42}$, $d = \frac{1}{231}$, $e = \frac{5}{21 \cdot 286}$, $f = \frac{1}{21 \cdot 286}$. The solution with $e = 1$ and $f = \frac{35}{6}$ does not survive the next level of Massey squares.

But the solution starting with $a = 1$, $b = \frac{1}{7}$ continues with $g = \frac{1}{29172}$, $h = \frac{1}{138567}$, $i = \frac{1}{646646}$, $j = \frac{5}{14872858}$, $k = \frac{1}{13520780}$. We will describe this family from a different point of view in Section 5.3, and we will show there (implicitly) that this solution survives to infinity.

Proposition 10. *In weight $l = 1$, there are exactly two non-equivalent true deformations.*

The problem of determining the explicit square zero cocycles in each weight l case seems to be a rich problem. We tried to say something about the rank of the finite Jacobimatrix either associated to the set of equations of type $M_{2rs} = 0$ for all r, s , or to those of type M_{23r} for all r ,

when we truncate the number of variables and consider only those equations involving these variables. With this matrix, it is obvious that the set of solutions (of the truncated problem) is an algebraic variety of dimension greater or equal to 1 (because the equations are homogeneous), but we could not decide whether the dimension drops down to 1 in each weight. In fact, within the possibilities of our computer, we computed (using all equations of type $M_{2rs} = 0$) the dimension of this variety as far as possible for $l = 2$ and it remained 2. Is the set of solutions always a variety? Is it always of finite dimension? Can we give asymptotics or bounds or a formula for the dimension?

5.3. Identifying the cocycles and the deformed algebras in weight zero

We now construct deformations from the previously determined weight l 2-cocycles given by their coefficients $a_{i,j}$ in the following way: using still the e_i for $i \geq 1$ as a basis, the deformed bracket reads

$$[e_i, e_j]_t = [e_i, e_j] + t a_{i,j} e_{i+j+l}.$$

It is clear that all square zero 2-cocycles give in this way true deformations of \mathfrak{m}_0 for which only the linear term is (possibly) non-zero; this means in particular that the bracket $[-, -]_t$ satisfies the Jacobi identity without adding terms containing higher powers in t .

The weight $l = 0$ case is the most interesting, because here deformations give automatically rise to \mathbb{N} -graded Lie algebras which must fit in the classification [1]. In this classification, the three \mathbb{N} -graded Lie algebras where e_1 has non-zero brackets with all other basis elements are

- (1) \mathfrak{m}_0 ; brackets: $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$,
- (2) \mathfrak{m}_2 ; brackets: $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+1}$ for all $j \geq 3$,
- (3) L_1 ; brackets: $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \geq 1$.

The complete set of infinitesimal deformations of L_1 and the complete set of formal deformations of L_1 is given in [2]. Let us consider in this section the same problem for \mathfrak{m}_0 in weight $l = 0$.

Taking as 2-cocycle the 2-family, we get in weight $l = 0$ a Lie algebra $\mathfrak{m}_0^1(t)$ which must be \mathbb{N} -graded and which is easily seen to be generated by e_1 and e_2 : indeed, $[e_1, e_i]_t = [e_1, e_i]$ for all $i \geq 2$. The complete relations for $\mathfrak{m}_0^1(t)$ are

$$\begin{cases} [e_1, e_i]_t = e_{i+1} & \forall i \geq 2, \\ [e_2, e_j]_t = t e_{j+2} & \forall j \geq 3. \end{cases}$$

Thus this family describes the deformation of \mathfrak{m}_0 to \mathfrak{m}_2 .

Now there is also a cocycle describing the deformation of \mathfrak{m}_0 to L_1 : observe that the generators e_i for $i \geq 1$ of \mathfrak{m}_0 do not satisfy the right relations, seen as elements of L_1 . Therefore, one must first perform a change of base: let us define $\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = e_3$, $\tilde{e}_4 = \frac{1}{2}e_4$ and in general

$$\tilde{e}_i = \frac{1}{(i-2)!} e_i$$

for all $i \geq 5$. Then the relations are easily computed to be

$$[\tilde{e}_1, \tilde{e}_i] = (i - 1)\tilde{e}_{i+1}.$$

The cocycle relation (2) for a cocycle given by coefficients $b_{i,j}$ transforms in the new basis to

$$(j - 1)b_{j+1,k} + (k - 1)b_{j,k+1} = (j + k + l - 1)b_{j,k}.$$

It is easy to check that the 2-cochain given by the coefficients $b_{i,j} = (j - i)$ is indeed a 2-cocycle for $l = 0$. One also easily checks that the 2-cocycle $b_{i,j} = (j - i)$ is of Massey square zero, i.e.

$$b_{i,j}b_{i+j,k} + b_{j,k}b_{j+k,i} + b_{k,i}b_{k+i,j} = 0.$$

It therefore determines a deformation $m_0^2(t)$ of m_0 in weight $l = 0$ to L_1 , and we showed in the previous section that these are all possible deformations of m_0 in weight 0.

Let us finish with the identification of the deformations in weight $l = 1$. It is clear that the 2-family leads to a non-trivial true deformation. This is then a weight 1 variant of the Lie algebra m_2 . The other cocycle, determined using MUPaD, is more interesting. Indeed, there is a general procedure of constructing positive weight, true deformations for m_0 : consider the Lie algebra L_1 , with its generators e_1, e_2, e_3 , etc., and its relations $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \geq 1$. Define a Lie algebra $L_1\{2\}$ by generators e_1, e_3, e_4 , etc. (the suppression of e_2 is indicated by $\{2\}$ in the notation!) and the relations of L_1 for the remaining generators. Introduce a new basis $f_1 := e_1, f_2 := e_3, f_3 := e_4$, etc., and another new basis $g_1 := e_1, g_k := (k - 1)!f_k$ for all $k > 1$. We compute the relations to

$$[g_1, g_k] = g_{k+1}, \quad [g_k, g_{k+1}] = \frac{k!(k-1)!}{(2k+1)!} g_{2k+2}.$$

When interpreted as a deformation of m_0 , one can then compute the coefficients $a_{i,j}$ of the corresponding 2-cocycle. One obtains $a_{2,3} = \frac{2!1!}{5!} = \frac{1}{60} \cdot 1$, $a_{3,4} = \frac{3!2!}{7!} = \frac{1}{60} \cdot \frac{1}{7}$, $a_{4,5} = \frac{4!3!}{9!} = \frac{1}{60} \cdot \frac{1}{42}$, $a_{5,6} = \frac{5!4!}{11!} = \frac{1}{60} \cdot \frac{1}{231}$, $a_{6,7} = \frac{6!5!}{13!} = \frac{1}{60} \cdot \frac{5}{21 \cdot 286}$, $a_{7,8} = \frac{7!6!}{15!} = \frac{1}{60} \cdot \frac{1}{21 \cdot 286}$, and so on. Thus, this deformation has as its infinitesimal cocycle the cocycle we determined using MUPaD before, up to a factor $\frac{1}{60}$. By construction, it is clear that it defines a cocycle and a true deformation, because $L_1\{2\}$ is a Lie algebra.

In the same way, one can define $L_1\{m\}$ by the span of the vectors e_1, e_{m+1}, e_{m+2} , etc., for any $m > 2$, in other words, by the suppression of all basis vectors from e_2 up to and including e_{m+1} . This gives a non-trivial true deformation of weight $m - 1$. Together with the deformation given by the 2-family, these two constitute two independent true deformations in any positive weight.

Acknowledgments

The work has been partially supported by the grants OTKA T034641 and T043034 and by the Erasmus program between Eötvös Loránd University Budapest and Université Louis Pasteur Strasbourg. Both authors are grateful to IHES where some of this work was accomplished, to Yury Nikolayevsky for useful remarks and to Matthias Borer who helped us to get hold on the weight $l > 0$ case by MUPaD based computations.

References

- [1] Alice Fialowski, Classification of graded Lie algebras with two generators, *Moscow Univ. Math. Bull.* 38 (2) (1983) 76–79.
- [2] Alice Fialowski, An example of formal deformations of Lie algebras, in: M. Hazewinkel, M. Gerstenhaber (Eds.), *Deformation Theory of Algebras and Structures and Applications*, Kluwer Academic Publishers, 1988, pp. 375–401.
- [3] Alice Fialowski, Dmitry Fuchs, Construction of miniversal deformations of Lie algebras, *J. Funct. Anal.* 161 (1999) 76–110.
- [4] Alice Fialowski, Dmitry Millionchshikov, Cohomology of graded Lie algebras of maximal class, *J. Algebra* 296 (1) (2006) 157–176.
- [5] Dmitry B. Fuchs, *Cohomology of Infinite Dimensional Lie Algebras*, Consultants Bureau, New York, 1987.
- [6] L.V. Gontcharova, Cohomology of Lie algebras of formal vector fields on the line, *Funct. Anal. Appl.* 7 (2) (1973) 6–14.
- [7] Aner Shalev, Efim I. Zelmanov, Narrow Lie algebras: A coclass theory and a characterization of the Witt algebra, *J. Algebra* 189 (1997) 294–331.
- [8] Michèle Vergne, Cohomologie des algèbres de Lie nilpotentes, *Bull. Soc. Math. France* 98 (1970) 81–116.